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Graded 1-Absorbing Primary Ideals

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Abstract. Let G be a group with identity e and R be a G-graded commutative ring with nonzero unity 1. In this article, we introduce the concept of graded 1-absorbing primary ideals. A proper graded ideal P of R is said to be a graded 1-absorbing primary ideal of R if whenever nonunit elements $x, y, z \in h(R)$ such that $xyz \in P$, then $xy \in P$ or $z \in \sqrt{P}$. Several properties of graded 1-absorbing primary ideals are investigated.

1. Introduction

Since graded prime ideals have a valuable performance in the theory of graded commutative rings, there are various procedures to generalize the concept of graded prime ideals. In [6], Naghani and Moghimi gave a generalization of graded prime ideals, called graded 2-absorbing ideals. A proper graded ideal P of R is said to be graded 2-absorbing if whenever $a, b, c \in h(R)$ such that $abc \in P$, then either $ab \in P$ or $ac \in P$ or $bc \in P$. Graded 2-absorbing ideals have been admirably studied in [2]. Graded primary ideals have been introduced and studied in [9]. A proper graded ideal P of R is said to be graded primary if for $x, y \in h(R)$ such that $xy \in P$, then either $x \in P$ or $y \in \sqrt{P}$. Recall from [10] that a proper graded ideal P of R is called a graded 2-absorbing primary ideal of R if whenever $a, b, c \in h(R)$ with $abc \in P$, then $ab \in P$ or $ac \in \sqrt{P}$ or $bc \in \sqrt{P}$.

In this article, we follow [4] to introduce and study the concept of graded 1-absorbing primary ideals of a graded commutative rings. A graded proper ideal P of a graded commutative ring R is said to be a graded 1-absorbing primary ideal of R if whenever nonunit elements $x, y, z \in h(R)$ such that $xyz \in P$, then $xy \in P$ or $z \in \sqrt{P}$. Among several results, we prove that the following implications hold and none of them is reversible:

graded primary ideal $\Rightarrow$ graded 1-absorbing primary ideal $\Rightarrow$ graded 2-absorbing primary ideal.

We prove that if P is a graded 1-absorbing primary ideal of a $\mathbb{Z}$-graded ring R, then $\sqrt{P}$ is a graded prime ideal of R (Proposition 2.4). We show that if P is a graded 1-absorbing primary ideal of R that is not graded primary, then there exist a homogeneous irreducible element $a \in R$ and a nonunit element $b \in h(R)$ such that $ab \in P$, but neither $a \in P$ nor $b \in \sqrt{P}$ (Proposition 2.11). We prove that if P is a graded 1-absorbing primary ideal of R, then $(P : a)$ is a graded primary ideal of R for every nonunit element $a \in h(R) - P$ (Proposition 2.13). We show that if P is a graded ideal of a $\mathbb{Z}$-graded divided ring R, then P is a graded 1-absorbing primary ideal of R if and only if P is a graded primary ideal of R (Proposition 2.16). In Proposition 2.20, we study graded 1-absorbing primary ideals under graded homomorphism. We close our article by proving that a proper graded ideal P is a graded 1-absorbing primary ideal of R if and only if whenever P_1, P_2 and P_3 are proper graded ideals of R such that $P_1P_2P_3 \subseteq P$, then either $P_1P_2 \subseteq P$ or $P_3 \subseteq \sqrt{P}$ (Corollary 2.23).

1.1. Preliminaries

Throughout this article, G will be a group with identity e and R a commutative ring with a nonzero unity 1. R is said to be G-graded if $R = \bigoplus_{g \in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$ where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. Also,

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we set $h(R) = \bigcup_{g\in G} R_g$. Moreover, it has been proved in [7] that R_e is a subring of R and $1 \in R_e$. Let I be an ideal of a graded ring R. Then I is said to be graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$ where $x_g \in I$ for all $g \in G$. An ideal of a graded ring need not be graded. Let R be a G-graded ring and I is a graded ideal of R. Then R/I is G-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$. If R and S are G-graded rings, then $R \times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$.

Lemma 1.1. *([5], Lemma 2.1) Let R be a G-graded ring.*

1. *If I and J are graded ideals of R, then $I + J$, IJ and $I \bigcap J$ are graded ideals of R.*
2. *If $x \in h(R)$, then Rx is a graded ideal of R.*

Let P be a proper graded ideal of R. Then the graded radical of P is $\sqrt{P}$, and is defined to be the set of all $r \in R$ such that for each $g \in G$, there exists a positive integer n_g satisfies $r_g^{n_g} \in P$. One can see that if $r \in h(R)$, then $r \in \sqrt{P}$ if and only if $r^n \in P$ for some positive integer n.

2. Graded 1-Absorbing Primary Ideals

In this section, we introduce and study the concept of graded 1-absorbing primary ideals.

Definition 2.1. *A proper graded ideal P of a graded ring R is said to be graded 1-absorbing primary if whenever nonunit elements $x, y, z \in h(R)$ such that $xyz \in P$, then $xy \in P$ or $z \in \sqrt{P}$.*

Clearly, every graded primary ideal is graded 1-absorbing primary ideal. The next example shows that the converse is not true in general.

Example 2.2. *Assume that R is trivially $\mathbb{Z}$-graded ring. Let K be a field and $R = K[X, Y]$ with $degX = 1 = degY$. Consider the graded ideal $P = (X^2, XY)$ of R. Then $\sqrt{P} = (X)$. Since for $X.Y.X \in P$, either $X.Y \in P$ or $X \in \sqrt{P}$, P is a graded 1-absorbing primary ideal of R. On the other hand, P is not graded primary ideal of R by ([10], Example 2.11).*

Also, it is clear that every graded 1-absorbing primary ideal is graded 2-absorbing primary ideal. The next example shows that the converse is not true in general.

Example 2.3. *Let $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider $P = \langle 12\rangle$. Then as $12 \in h(R)$, P is a graded ideal of R. By ([3], Example 2.2 (ii)), P is a graded 2-absorbing primary ideal of R. On the other hand, $2, 3 \in h(R)$ such that $2.2.3 \in P$, but neither $2.2 \in P$ nor $3 \in \sqrt{P}$. So, P is not graded 1-absorbing primary ideal of R.*

If P is a graded ideal of a G-graded ring R, then $\sqrt{P}$ need not to be a graded ideal of R; see ([8], Exercises 17 and 13 on pp. 127-128). However, in ([1], Lemma 2.13), it has been proved that if P is a graded ideal of a $\mathbb{Z}$-graded ring R, then $\sqrt{P}$ is a graded ideal of R.

Proposition 2.4. *Let R be a $\mathbb{Z}$-graded ring and P be a graded ideal of R. If P is a graded 1-absorbing primary ideal of R, then $\sqrt{P}$ is a graded prime ideal of R.*

Proof. Let $a, b \in h(R)$ such that $ab \in \sqrt{P}$. We may assume that a, b are nonunit elements of R. Let $k \geq 2$ be an even positive integer such that $(ab)^k \in P$. Then $k = 2s$ for some positive integer $s \geq 1$. Since $(ab)^k = a^k b^k = a^s a^s b^k \in P$ and P is a graded 1-absorbing primary ideal of R, we conclude that $a^s a^s = a^k \in P$ or $b^k \in P$. Hence, $a \in \sqrt{P}$ or $b \in \sqrt{P}$. Thus $\sqrt{P}$ is a graded prime ideal of R. □

Definition 2.5. *Let R be a G-graded ring and P be a graded ideal of R. Assume that $g \in G$ such that $P_g \neq R_g$. Then P is said to be a g-1-absorbing primary ideal of R if whenever nonunit elements $x, y, z \in R_g$ such that $xyz \in P$, then $xy \in P$ or $z \in \sqrt{P}$.*

Proposition 2.6. *Let R be a G-graded ring and $g \in G$. If R has a g-1-absorbing primary ideal that is not a g-primary ideal, then the sum of every nonunit element of R_g and every unit element of R_g is a unit element of R_g.*

Proof. Suppose that P is a g-1-absorbing primary ideal of R that is not a g-primary ideal of R. Hence, there exist nonunit elements $a, b \in R_g$ such that neither $a \in P$ nor $a \in \sqrt{P}$. Let w be a nonunit element of R_g. Since $wab \in P$ and P is a g-1-absorbing primary ideal of R and $b \notin \sqrt{P}$, we conclude that $wa \in P$. Let u be a unit element of R_g. Suppose that $w+u$ is a nonunit element of R_g. Since $(w+u)ab \in P$ and P is a g-1-absorbing primary ideal of R_g and $b \notin \sqrt{P}$, we conclude that $(w+u)a = wa + ua \in P$. Since $wa \in P$, we conclude that $a \in P$, which is a contradiction. Thus, $w+u$ is a unit element of R_g. □

Corollary 2.7. *Let R be a G-graded ring. If R has an e-1-absorbing primary ideal that is not an e-primary ideal, then R_e is a quasilocal ring (R_e has exactly one maximal ideal).*

Proof. By Proposition 2.6, the sum of every nonunit element of R_e and every unit element of R_e is a unit element of R_e, and then by ([4], Lemma 1), R_e is a quasilocal ring. □

Also, in view of Proposition 2.6, we have the following conclusion.

Corollary 2.8. *Let R be a G-graded ring and $g \in G$. If R_g has a nonunit element and a unit element whose sum is nonunit element in R_g, then a graded ideal P of R is a g-1-absorbing primary ideal of R if and only if P is a g-primary ideal of R.*

In view of Corollary 2.8, we have the following result.

Proposition 2.9. *Let $R = S \times T$, where S and T are G-graded commutative rings with a nonzero unity 1. Suppose that P is a graded ideal of R and $g \in G$. Then the following assertions are equivalent:*

1. *P is a g-1-absorbing primary ideal of R.*
2. *P is a g-primary ideal of R.*
3. *$P = I \times T$ for some g-primary ideal I of S or $P = S \times J$ for some g-primary ideal J of T.*

Proof. In view of Corollary 2.8, in particular, a graded ideal P of R is a g-1-absorbing primary ideal of R if and only if P is a g-primary ideal of R, and it is familiar that P is a g-primary ideal of R if and only if $P = I \times T$ for some g-primary ideal I of S or $P = S \times J$ for some g-primary ideal J of T. So, the result holds. □

Definition 2.10. *Let R be a graded ring. Then $x \in h(R)$ is said to be a homogeneous reducible element of R if $x = yz$ for some nonunit elements $y, z \in h(R)$. Otherwise, x is called a homogeneous irreducible element of R.*

Proposition 2.11. *Let R be a graded ring. Suppose that P is a graded 1-absorbing primary ideal of R that is not a graded primary ideal of R. Then there exist a homogeneous irreducible element $a \in R$ and a nonunit element $b \in h(R)$ such that $ab \in P$, but neither $a \in P$ nor $b \in \sqrt{P}$. Moreover, if $xy \in P$ for some nonunit elements $x, y \in h(R)$ such that neither $x \in P$ nor $y \in \sqrt{P}$, then x is a homogeneous irreducible element of R.*

Proof. Since P is not a graded primary ideal of R, there exist nonunit elements $a, b \in h(R)$ such that $ab \in P$ and neither $a \in P$ nor $b \in \sqrt{P}$. Suppose that a is a homogeneous reducible element of R. Then $a = cd$ for some nonunit elements $c, d \in h(R)$. Since $ab = cdb \in P$ and P is a graded

1-absorbing primary ideal of R and $b \in \sqrt{P}$, we achieve that $a = cd \in P$, which is a contradiction. Thus, a is a homogeneous irreducible element of R. $\square$

Lemma 2.12. *Let R be a G-graded ring and P be a graded ideal of R. Then $(P : a) = \{x \in R : xa \in P\}$ is a graded ideal of R for every $a \in h(R)$.*

Proof. Let $a \in h(R)$. Then it is clear that $(P : a)$ is an ideal of R. Let $x \in (P : a)$. Then $x \in R$ such that $xa \in P$. Now, $x = \sum_{g \in G} x_g$ where $x_g \in R_g$ for all $g \in G$. So, $x_g a \in h(R)$ for all $g \in G$ with $\sum_{g \in G} x_g a = \left(\sum_{g \in G} x_g\right) a = xa \in P$, and since P is a graded ideal of R, we conclude that $x_g a \in P$ for all $g \in G$, which implies that $x_g \in (P : a)$ for all $g \in G$. Hence, $(P : a)$ is a graded ideal of R. $\square$

Proposition 2.13. *Let R be a graded ring R and P be a graded ideal of R. If P is a graded 1-absorbing primary ideal of R, then $(P : a)$ is a graded primary ideal of R for every nonunit element $a \in h(R) - P$.*

Proof. Let $a \in h(R) - P$ such that a is a nonunit element. Then by Lemma 2.12, $(P : a)$ is a graded ideal of R. Assume that $x, y \in h(R)$ such that $xy \in (P : a)$. We may assume that x, y are nonunit elements of R. Suppose that $x \notin (P : a)$. Then $xa \notin P$. Since $axy \in P$ and P is a graded 1-absorbing primary ideal of R and $ax \notin P$, we achieve that $y \in \sqrt{P} \subseteq \sqrt{(P : a)}$. Thus, $(P : a)$ is a graded primary ideal of R. $\square$

Proposition 2.14. *Let R be a $\mathbb{Z}$-graded ring and P be a graded 1-absorbing primary ideal of R. Then for every nonunit element $a \in h(R) - P$, we have either*

$$P \subsetneqq (P : a) \text{ or } \sqrt{(P : a)} = \sqrt{P}$$

Proof. Let $a \in h(R) - P$ be a nonunit element. Clearly, $P \subseteq (P : a)$. If $a \in \sqrt{P}$, then $a^n \in P$ for some positive integer n. We may assume that n is the least positive integer such that $a^n \in P$. Then $a^{n-1} \in (P : a) - P$, and hence $P \subsetneqq (P : a)$. Suppose that $a \notin \sqrt{P}$. Let $x \in (P : a)$. Then by Lemma 2.12, $x_i \in (P : a)$ for all $i \in \mathbb{Z}$. Now, for any $i \in \mathbb{Z}$, $ax_i \in P \subseteq \sqrt{P}$, and since $\sqrt{P}$ is a graded prime ideal of R by Proposition 2.4 and $a \notin \sqrt{P}$, we conclude that $x_i \in \sqrt{P}$ for all $i \in \mathbb{Z}$, which implies that $x \in \sqrt{P}$. Hence, $P \subseteq (P : a) \subseteq \sqrt{P}$, which implies that $\sqrt{P} \subseteq \sqrt{(P : a)} \subseteq \sqrt{P}$. Thus, $\sqrt{(P : a)} = \sqrt{P}$. $\square$

Definition 2.15. *Let R be a graded ring.*

1. *For $a, b \in h(R)$, we say that a divides b (we write $a|b$) if $b = ax$ for some $x \in h(R)$.*
2. *R is said to be a graded chained ring if for every $a, b \in h(R)$, we have either $a|b$ or $b|a$.*
3. *R is said to be a graded divided ring if for every graded prime ideal P of R and for every $a \in h(R) - P$, we have $a|p$ for every $p \in P$.*

Clearly, every graded chained ring is a graded divided ring.

Proposition 2.16. *Let R be a $\mathbb{Z}$-graded divided ring and P be a graded ideal of R. Then P is a graded 1-absorbing primary ideal of R if and only if P is a graded primary ideal of R.*

Proof. Suppose that P is a graded 1-absorbing primary ideal of R. Let $a, b \in h(R)$ such that $ab \in P$ and $b \notin \sqrt{P}$. We may assume that a, b are nonunit elements of R. Since $\sqrt{P}$ is a graded prime ideal of R by Proposition 2.4 and $b \notin \sqrt{P}$, we have that $a \in \sqrt{P}$. Since R is a graded divided ring, we have that $b|a$, which means that $a = bw$ for some $w \in h(R)$. Since $b \notin \sqrt{P}$ and $a \in \sqrt{P}$, we achieve that w is a nonunit element of R. Since $ab = bwb \in P$ and P is a graded 1-absorbing

primary ideal of R and $b \notin \sqrt{P}$, we have that $a = bw \in P$. Thus, P is a graded primary ideal of R. The converse is clear. □

Corollary 2.17. *Let R be a $\mathbb{Z}$-graded chained ring and P be a graded ideal of R. Then P is a graded 1-absorbing primary ideal of R if and only if P is a graded primary ideal of R.*

Proposition 2.18. *Let R be a graded divided integral domain and P be a graded prime ideal of R. Then P^n is a graded primary ideal of R for every positive integer n, and hence P^n is a graded 1-absorbing primary ideal of R for every positive integer n.*

Proof. Let n be a positive integer. If $n = 1$, then it is clear. Suppose that $n \geq 2$. Then by Lemma 1.1, P^n is a graded ideal of R. Let $a, b \in h(R)$ such that $ab \in P^n$. Then $ab = p_1x_1 + p_2x_2 + ... + p_kx_k \in P^n$ for some $p_1, p_2, ..., p_k \in P$ and $x_1, x_2, ..., x_k \in P^{n-1}$ for some positive integer k. Suppose that $b \notin P$. Then since R is a graded divided ring, we have that $b|p_i$ for all $1 \leq i \leq k$, which means that $p_i = c_ib$ for some $c_i \in h(R) \bigcap P$, which implies that $ab = c_1bx_1 + c_2bx_2 + ... + c_kbx_k$, and then $b(a - (c_1x_1 + c_2x_2 + ... + c_kx_k)) = 0$. Since R is an integral domain, we have that $a = c_1x_1 + c_2x_2 + ... + c_kx_k \in P^n$. Hence, P^n is a graded primary ideal of R. □

Proposition 2.19. *Let R be a graded ring and $P_1, P_2, ..., P_n$ be graded 1-absorbing primary ideals of R. If $\sqrt{P_i} = \sqrt{P_j} = Q$ for every i, j, then $P = \bigcap_{i=1}^{n} P_i$ is a graded 1-absorbing primary ideal of R.*

Proof. Suppose that $x, y, z \in h(R)$ are nonunit elements such that $xyz \in P$. Suppose that $xy \notin P$. Then $xy \notin P_k$ for some $1 \leq k \leq n$. Since P_k is a graded 1-absorbing primary ideal of R and $xyz \in P_k$ and $xy \notin P_k$, we have that $z \in \sqrt{P_k} = Q = \sqrt{P}$. Hence, P is a graded 1-absorbing primary ideal of R. □

Let R and S be two G-graded rings. A ring homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$.

Proposition 2.20. *Let R and S be G-graded rings and $f : R \to S$ be a graded homomorphism such that $f(1_R) = 1_S$. Then the following hold:*

1. *If K is a graded 1-absorbing primary ideal of S and $f(x)$ is a nonunit element of S for every nonunit element x of R, then $f^{-1}(K)$ is a graded 1-absorbing primary ideal of R.*
2. *If P is a graded 1-absorbing primary ideal of R and f is surjective with $Ker(f) \subseteq P$, then $f(P)$ is a graded 1-absorbing primary ideal of S.*

Proof.
1. Clearly, $f^{-1}(K)$ is a graded ideal of R. Let $x, y, z \in h(R)$ be nonunit elements such that $xyz \in f^{-1}(K)$. Then $f(x), f(y), f(z) \in h(S)$ are nonunit elements such that $f(x)f(y)f(z) = f(xyz) \in K$. Since K is a graded 1-absorbing primary ideal of S, we have that $f(xy) = f(x)f(y) \in K$ or $f(z) \in \sqrt{K}$, which implies that $xy \in f^{-1}(K)$ or $z \in f^{-1}(\sqrt{K}) = \sqrt{f^{-1}(K)}$. Thus, $f^{-1}(K)$ is a graded 1-absorbing primary ideal of R.
2. Clearly, $f(P)$ is a graded ideal of S. Let $a, b, c \in h(S)$ be nonunit elements such that $abc \in f(P)$. Then since f is surjective, there exist nonunit elements $x, y, z \in h(R)$ such that $f(x) = a, f(y) = b$ and $f(z) = c$. Now, $f(xyz) = f(x)f(y)f(z) = abc \in f(P)$. Since $Ker(f) \subseteq P$, we have that $xyz \in P$. Since P is a graded 1-absorbing primary ideal of R, we have that $xy \in P$ or $z \in \sqrt{P}$, which implies that $ab = f(x)f(y) = f(xy) \in f(P)$ or $c = f(z) \in f(\sqrt{P}) = \sqrt{f(P)}$ as f is surjective and $Ker(f) \subseteq P$. Hence, $f(P)$ is a graded 1-absorbing primary ideal of S.

□

Corollary 2.21. *Let P and K be proper graded ideals of a graded ring R with $K \subseteq P$. If $U(R/K) = \{a + K : a \in U(R)\}$, then P is a graded 1-absorbing primary ideal of R if and only if P/K is a graded 1-absorbing primary ideal of R/K.*

Proof. Let $f : R \to R/K$ such that $f(x) = x + K$. Then f is surjective graded homomorphism and $f(1_R) = 1_{R/K}$. Suppose that P is a graded 1-absorbing primary ideal of R. Since f is surjective and $Ker(f) = K \subseteq P$, by Proposition 2.20(2), we have that $f(P) = P/K$ is a graded 1-absorbing primary ideal of R/K. Conversely, $f^{-1}(P/K) = P$ is a graded 1-absorbing primary ideal of R by Proposition 2.20(1). □

Proposition 2.22. *Let P be a graded 1-absorbing primary ideal of a G-graded ring R and K be a proper graded ideal of R. If $x, y \in h(R)$ are nonunit elements such that $xyK \subseteq P$, then either $xy \in P$ or $K \subseteq \sqrt{P}$.*

Proof. Suppose that $xy \notin P$. Let $a \in K$. Then since K is a proper graded ideal of R, we have that $a_g \in K$ is a nonunit element for all $g \in G$. Now, for any $g \in G$, $xya_g \in P$. Since P is a graded 1-absorbing primary ideal of R and $xy \notin P$, we achieve that $a_g \in \sqrt{P}$ for all $g \in G$, which implies that $a \in \sqrt{P}$. Hence, $K \subseteq \sqrt{P}$. □

Corollary 2.23. *Let P be a proper graded ideal of a G-graded ring R. Then P is a graded 1-absorbing primary ideal of R if and only if whenever P_1, P_2 and P_3 are proper graded ideals of R such that $P_1P_2P_3 \subseteq P$, then either $P_1P_2 \subseteq P$ or $P_3 \subseteq \sqrt{P}$.*

Proof. Suppose that P is a graded 1-absorbing primary ideal of R. Let P_1, P_2 and P_3 be proper graded ideals of R such that $P_1P_2P_3 \subseteq P$ and $P_1P_2 \nsubseteq P$. Then there exist $x \in P_1$ and $y \in P_2$ such that $xy \notin P$, and then there exist $g, h \in G$ such that $x_g y_h \notin P$. Since P_1 and P_2 are proper graded ideals of R, we achieve that $x_g \in P_1$ is a nonunit element and $y_h \in P_2$ is a nonunit element too. Since $x_g y_h P_3 \subseteq P$ and $x_g y_h \notin P$, we have that $P_3 \subseteq \sqrt{P}$ by Proposition 2.22. Conversely, let $x, y, z \in h(R)$ be nonunit elements such that $xyz \in P$. Then $P_1 = \langle x \rangle$, $P_2 = \langle y \rangle$ and $P_3 = \langle z \rangle$ are proper graded ideals of R such that $P_1P_2P_3 \subseteq P$, and then by assumption, we have either $P_1P_2 \subseteq P$ or $P_3 \subseteq \sqrt{P}$, which implies that either $xy \in P$ or $z \in \sqrt{P}$. Hence, P is a graded 1-absorbing primary ideal of R. □

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